A COMPLETELY MONOTONIC FUNCTION INVOLVING THE TRI- AND TETRA-GAMMA FUNCTIONS

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Abstract. The psi function $\psi(x)$ is defined by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ and $\psi^{(i)}(x)$ for $i\in\mathbb{N}$ denote the polygamma functions, where $\Gamma(x)$ is the gamma function. In this paper we prove that a function involving the difference between $[\psi'(x)]^2+\psi''(x)$ and a proper fraction of x is completely monotonic on $(0,\infty)$.

1. Introduction

We recall [3, Chapter XIII] and [4, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x)\ge 0 \tag{1}$$

for $x\in I$ and $n\ge 0$.

The famous Bernstein-Widder Theorem (see [4, p. 160, Theorem 12a]) states that a function $f(x)$ on $[0,\infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $\alpha(t)$ such that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\alpha(t) \tag{2}$$

converges for $x\in[0,\infty)$. This says that a completely monotonic function $f(x)$ on $[0,\infty)$ is a Laplace transform of the measure $\alpha(t)$.

We also recall that the classical Euler gamma function $\Gamma(x)$ is defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}\,t,\quad x>0. \tag{3}$$

2000 *Mathematics Subject Classification.* Primary 26A48, 33B15; Secondary 26A51, 26D07, 26D10.

Key words and phrases. Completely monotonic function, polygamma function, inequality.

The authors were supported partially by the Science Foundation of Tianjin Polytechnic University.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or di-gamma function, and the derivatives $\psi^{(i)}(x)$ for $i\in\mathbb{N}$ are respectively called the polygamma functions. In particular, the functions $\psi'(x)$ and $\psi''(x)$ are called the tri-gamma and tetra-gamma functions.

In [2, p. 208, (4.39)], it was established that the inequality

$$[\psi'(x)]^2+\psi''(x)>\frac{p(x)}{900x^4(x+1)^{10}} \tag{4}$$

holds for $x>0$, where

$$\begin{aligned} p(x)&=75x^{10}+900x^9+4840x^8+15370x^7+31865x^6+45050x^5\\ &\quad+44101x^4+29700x^3+13290x^2+3600x+450. \end{aligned} \tag{5}$$

The aim of this paper is to prove the complete monotonicity of the difference between two functions on both sides of the inequality (4).

Our main result may be stated as the following theorem.

Theorem 1. *The function*

$$g(x)=[\psi'(x)]^2+\psi''(x)-\frac{p(x)}{900x^4(x+1)^{10}} \tag{6}$$

is completely monotonic on $(0,\infty)$*, where the function* $p(x)$ *is defined by* (5).

Remark 1. By the definition of completely monotonic functions, it is easy to see that Theorem 1 is stronger than the inequality (4), so Theorem 1 generalizes the inequality (4).

2. Proof of Theorem 1

By the recursion formula [1, pp. 258 and 260, 6.3.5 and 6.4.6]

$$\psi^{(n-1)}(x+1)=\psi^{(n-1)}(x)+\frac{(-1)^{n-1}(n-1)!}{x^n} \tag{7}$$

for $x>0$ and $n\in\mathbb{N}$, direct calculation produces

$$\begin{gathered} g(x)-g(x+1)=\bigl[\psi'(x)-\psi'(x+1)\bigr]\bigl[\psi'(x)+\psi'(x+1)\bigr]\\ +\bigl[\psi''(x)-\psi''(x+1)\bigr]-\biggl[\frac{p(x)}{900x^4(x+1)^{10}}-\frac{p(x+1)}{900(x+1)^4(x+2)^{10}}\biggr] \end{gathered}$$

$$=\frac{1}{x^2}\biggl[2\psi'(x)-\frac{1}{x^2}\biggr]-\frac{2}{x^3}-\biggl[\frac{p(x)}{900x^4(x+1)^{10}}-\frac{p(x+1)}{900(x+1)^4(x+2)^{10}}\biggr]$$

$$
\begin{aligned}
&= \frac{2}{x^2}\left[\psi'(x) - \frac{1}{2x^2} - \frac{1}{x} - \frac{p(x)}{1800x^2(x+1)^{10}} + \frac{x^2p(x+1)}{1800(x+1)^4(x+2)^{10}}\right]\\
&= \frac{2}{x^2}\left[\psi'(x) - \frac{1}{2x} - \frac{3}{4x^2} - \frac{28}{5(x+1)} + \frac{51}{10(x+2)} + \frac{251}{120(x+1)^2}\right.\\
&\quad + \frac{331}{120(x+2)^2} - \frac{7}{6(x+1)^3} + \frac{17}{12(x+2)^3} + \frac{13}{90(x+1)^4} + \frac{49}{72(x+2)^4}\\
&\quad - \frac{13}{180(x+1)^5} + \frac{47}{180(x+2)^5} - \frac{1}{120(x+1)^6} - \frac{1}{60(x+2)^6}\\
&\quad + \frac{1}{180(x+1)^7} - \frac{2}{45(x+2)^7} + \frac{1}{200(x+1)^8} - \frac{13}{600(x+2)^8}\\
&\quad \left. + \frac{1}{900(x+1)^9} - \frac{1}{450(x+2)^9} - \frac{1}{1800(x+1)^{10}} + \frac{1}{450(x+2)^{10}}\right]\\
&\triangleq \frac{2}{x^2}H(x).
\end{aligned}
$$

Using the formula [1, p. 255, 6.1.1]

$$
\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}\,\mathrm{d}\,t \tag{8}
$$

for $r > 0$ and $x > 0$ and the integral representations [1, p. 260, 6.4.1]

$$
\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}\,t \tag{9}
$$

for $n \in \mathbb{N}$ and $x > 0$ gives

$$
\begin{aligned}
H(x) = \int_0^\infty \biggl(&\frac{t}{1-e^{-t}} - \frac{1}{2} - \frac{3}{4}t - \frac{28}{5}e^{-t} + \frac{51}{10}e^{-2t} + \frac{251}{120}te^{-t}\\
&+ \frac{331}{120}te^{-2t} - \frac{7}{12}t^2e^{-t} + \frac{17}{24}t^2e^{-2t} + \frac{13}{540}t^3e^{-t} + \frac{49}{432}t^3e^{-2t}\\
&- \frac{13}{4320}t^4e^{-t} + \frac{47}{4320}t^4e^{-2t} - \frac{1}{14400}t^5e^{-t} - \frac{1}{7200}t^5e^{-2t}\\
&+ \frac{1}{129600}t^6e^{-t} - \frac{2}{32400}t^6e^{-2t} + \frac{1}{1008000}t^7e^{-t} - \frac{13}{3024000}t^7e^{-2t}\\
&+ \frac{1}{36288000}t^8e^{-t} - \frac{1}{18144000}t^8e^{-2t} - \frac{1}{653184000}t^9e^{-t}\\
&+ \frac{1}{163296000}t^9e^{-2t}\biggr)e^{-xt}\,\mathrm{d}\,t\\
= \frac{1}{653184000}&\int_0^\infty \frac{1}{e^t-1}\bigl[163296000(t-2)e^{3t} - e^{2t}\bigl(t^9 - 18t^8\\
&- 648t^7 - 5040t^6 + 45360t^5 + 1965600t^4 - 15724800t^3\\
&+ 381024000t^2 - 1856131200t + 3331238400\bigr) + e^t\bigl(5t^9 - 54t^8\\
&- 3456t^7 - 45360t^6 - 45360t^5 + 9072000t^4 + 58363200t^3
\end{aligned}
$$

$$
\begin{aligned}
&+843696000t^2+435456000t+6989068800\bigr)-4\bigl(t^9-9t^8\\
&-702t^7-10080t^6-22680t^5+1776600t^4+18522000t^3\\
&+115668000t^2+450424800t+832809600\bigr)\bigr]e^{-(x+2)t}\,\mathrm{d}\,t\\
\triangleq{}&\frac{1}{653184000}\int_0^\infty\frac{1}{e^t-1}\theta(t)e^{-(x+2)t}\,\mathrm{d}\,t.
\end{aligned}
$$

Straightforward computation yields

$$
\begin{aligned}
\theta'(t)&=163296000(3t-5)e^{3t}-\bigl(4806345600-2950214400t\\
&\quad+714873600t^2-23587200t^3+4158000t^4+60480t^5-14616t^6\\
&\quad-1440t^7-27t^8+2t^9\bigr)e^{2t}+\bigl(7424524800+2122848000t\\
&\quad+1018785600t^2+94651200t^3+8845200t^4-317520t^5-69552t^6\\
&\quad-3888t^7-9t^8+5t^9\bigr)e^t-36\bigl(50047200+25704000t+6174000t^2\\
&\quad+789600t^3-12600t^4-6720t^5-546t^6-8t^7+t^8\bigr),\\
\theta''(t)&=489888000(3t-4)e^{3t}-4\bigl(1665619200-1117670400t\\
&\quad+339746400t^2-7635600t^3+2154600t^4+8316t^5-9828t^6\\
&\quad-774t^7-9t^8+t^9\bigr)e^{2t}+\bigl(9547372800+4160419200t\\
&\quad+1302739200t^2+130032000t^3+7257600t^4-734832t^5\\
&\quad-96768t^6-3960t^7+36t^8+5t^9\bigr)e^t-144\bigl(6426000+3087000t\\
&\quad+592200t^2-12600t^3-8400t^4-819t^5-14t^6+2t^7\bigr),\\
\theta^{(3)}(t)&=4408992000(t-1)e^{3t}-4\bigl(2213568000-1555848000t\\
&\quad+656586000t^2-6652800t^3+4350780t^4-42336t^5-25074t^6\\
&\quad-1620t^7-9t^8+2t^9\bigr)e^{2t}+\bigl(13707792000+6765897600t\\
&\quad+1692835200t^2+159062400t^3+3583440t^4-1315440t^5\\
&\quad-124488t^6-3672t^7+81t^8+5t^9\bigr)e^t-1008\bigl(441000+169200t\\
&\quad-5400t^2-4800t^3-585t^4-12t^5+2t^6\bigr),\\
\theta^{(4)}(t)&=4408992000(3t-2)e^{3t}-16\bigl(717822000-449631000t\\
&\quad+323303400t^2+1024380t^3+2122470t^4-58779t^5-15372t^6\\
&\quad-828t^7+t^9\bigr)e^{2t}+\bigl(20473689600+10151568000t+2170022400t^2
\end{aligned}
$$

$$
\begin{aligned}
&+173396160t^3-2993760t^4-2062368t^5-150192t^6-3024t^7\\
&+126t^8+5t^9\big)e^t-12096\big(14100-900t-1200t^2-195t^3-5t^4+t^5\big),\\
\theta^{(5)}(t)={}&13226976000(3t-1)e^{3t}-16\big(986013000-252655200t\\
&+649679940t^2+10538640t^3+3951045t^4-209790t^5\\
&-36540t^6-1656t^7+9t^8+2t^9\big)e^{2t}+\big(30625257600\\
&+14491612800t+2690210880t^2+161421120t^3-13305600t^4\\
&-2963520t^5-171360t^6-2016t^7+171t^8+5t^9\big)e^t\\
&-60480\big(-180-480t-117t^2-4t^3+t^4\big),\\
\theta^{(6)}(t)={}&119042784000te^{3t}-64\big(429842700+198512370t\\
&+332743950t^2+9220365t^3+1713285t^4-159705t^5\\
&-21168t^6-810t^7+9t^8+t^9\big)e^{2t}+\big(45116870400\\
&+19872034560t+3174474240t^2+108198720t^3-28123200t^4\\
&-3991680t^5-185472t^6-648t^7+216t^8+5t^9\big)e^t\\
&-120960\big(-240-117t-6t^2+2t^3\big),\\
\theta^{(7)}(t)={}&119042784000(1+3t)e^{3t}-64\big(1058197770+1062512640t\\
&+693148995t^2+25293870t^3+2628045t^4-446418t^5-48006t^6\\
&-1548t^7+27t^8+2t^9\big)e^{2t}+\big(64988904960+26220983040t\\
&+3499070400t^2-4294080t^3-48081600t^4-5104512t^5\\
&-190008t^6+1080t^7+261t^8+5t^9\big)e^t-362880\big(2t^2-39-4t\big),\\
\theta^{(8)}(t)={}&357128352000(2+3t)e^{3t}-128\big(1589454090+1755661635t\\
&+731089800t^2+30549960t^3+1512000t^4-590436t^5-53424t^6\\
&-1440t^7+36t^8+2t^9\big)e^{2t}+\big(91209888000+33219123840t\\
&+3486188160t^2-196620480t^3-73604160t^4-6244560t^5\\
&-182448t^6+3168t^7+306t^8+5t^9\big)e^t-1451520(t-1),\\
\theta^{(9)}(t)={}&3214155168000(1+t)e^{3t}-128\big(4934569815+4973502870t\\
&+1553829480t^2+67147920t^3+71820t^4-1501416t^5-116928t^6
\end{aligned}
$$

$$
\begin{aligned}
&- 2592t^7 + 90t^8 + 4t^9\big)e^{2t} + \big(124429011840 + 40191500160t\\
&+ 2896326720t^2 - 491037120t^3 - 104826960t^4 - 7339248t^5\\
&- 160272t^6 + 5616t^7 + 351t^8 + 5t^9\big)e^t - 1451520,\\
\theta^{(10)}(t) &= e^t\big[164620512000 + 45984153600t + 1423215360t^2 - 910344960t^3\\
&- 141523200t^4 - 8300880t^5 - 120960t^6 + 8424t^7 + 396t^8 + 5t^9\\
&- 512\big(3710660625 + 3263666175t + 827275680t^2 + 33645780t^3\\
&- 1840860t^4 - 926100t^5 - 63000t^6 - 1116t^7 + 54t^8 + 2t^9\big)e^t\\
&+ 3214155168000(4 + 3t)e^{2t}\big]\\
&\triangleq e^t\theta_1(t),\\
\theta_1'(t) &= 3214155168000(11 + 6t)e^{2t} - 512\big(6974326800 + 4918217535t\\
&+ 928213020t^2 + 26282340t^3 - 6471360t^4 - 1304100t^5 - 70812t^6\\
&- 684t^7 + 72t^8 + 2t^9\big)e^t + 9\big(5109350400 + 316270080t\\
&- 303448320t^2 - 62899200t^3 - 4611600t^4 - 80640t^5\\
&+ 6552t^6 + 352t^7 + 5t^8\big),\\
\theta_1''(t) &= 8\big[1607077584000(7 + 3t)e^{2t} - 64\big(11892544335 + 6774643575t\\
&+ 1007060040t^2 + 396900t^3 - 12991860t^4 - 1728972t^5 - 75600t^6\\
&- 108t^7 + 90t^8 + 2t^9\big)e^t + 9\big(39533760 - 75862080t - 23587200t^2\\
&- 2305800t^3 - 50400t^4 + 4914t^5 + 308t^6 + 5t^7\big)\big],\\
\theta_1^{(3)}(t) &= 8\big[1607077584000(17 + 6t)e^{2t} - 64\big(18667187910 + 8788763655t\\
&+ 1008250740t^2 - 51570540t^3 - 21636720t^4 - 2182572t^5\\
&- 76356t^6 + 612t^7 + 108t^8 + 2t^9\big)e^t - 63\big(10837440 + 6739200t\\
&+ 988200t^2 + 28800t^3 - 3510t^4 - 264t^5 - 5t^6\big)\big],\\
\theta_1^{(4)}(t) &= 16\big[3214155168000(10 + 3t)e^{2t} - 32\big(27455951565\\
&+ 10805265135t + 853539120t^2 - 138117420t^3 - 32549580t^4\\
&- 2640708t^5 - 72072t^6 + 1476t^7 + 126t^8 + 2t^9\big)e^t\\
&- 945\big(224640 + 65880t + 2880t^2 - 468t^3 - 44t^4 - t^5\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
\theta_1^{(5)}(t) &= 16\big[3214155168000(23+6t)e^{2t}-32\big(38261216700\\
&\quad+12512343375t+439186860t^2-268315740t^3-45753120t^4\\
&\quad-3073140t^5-61740t^6+2484t^7+144t^8+2t^9\big)e^t\\
&\quad-945\big(65880+5760t-1404t^2-176t^3-5t^4\big)\big],\\
\theta_1^{(6)}(t) &= 64\big[3214155168000(13+3t)e^{2t}-8\big(50773560075\\
&\quad+13390717095t-365760360t^2-451328220t^3-61118820t^4\\
&\quad-3443580t^5-44352t^6+3636t^7+162t^8+2t^9\big)e^t\\
&\quad-945\big(1440-702t-132t^2-5t^3\big)\big],\\
\theta_1^{(7)}(t) &= 64\big[3214155168000(29+6t)e^{2t}-8\big(64164277170\\
&\quad+12659196375t-1719745020t^2-695803500t^3-78336720t^4\\
&\quad-3709692t^5-18900t^6+4932t^7+180t^8+2t^9\big)e^t\\
&\quad+2835\big(234+88t+5t^2\big)\big],\\
\theta_1^{(8)}(t) &= 128\big[6428310336000(16+3t)e^{2t}-4\big(76823473545\\
&\quad+9219706335t-3807155520t^2-1009150380t^3-96885180t^4\\
&\quad-3823092t^5+15624t^6+6372t^7+198t^8+2t^9\big)e^t+2835(44+5t)\big],\\
\theta_1^{(9)}(t) &= 128\big[6428310336000(35+6t)e^{2t}-4\big(86043179880\\
&\quad+1605395295t-6834606660t^2-1396691100t^3-116000640t^4\\
&\quad-3729348t^5+60228t^6+7956t^7+216t^8+2t^9\big)e^t+14175\big],\\
\theta_1^{(10)}(t) &= 512e^t\big[6428310336000(19+3t)e^t-87648575175+12063818025t\\
&\quad+11024679960t^2+1860693660t^3+134647380t^4+3367980t^5\\
&\quad-115920t^6-9684t^7-234t^8-2t^9\big]\\
&\triangleq 512e^t\theta_2(t),\\
\theta_2'(t) &= 9\big[1340424225+2449928880t+620231220t^2+59843280t^3\\
&\quad+1871100t^4-77280t^5-7532t^6-208t^7-2t^8\\
&\quad+714256704000(22+3t)e^t\big],\\
\theta_2''(t) &= 72\big[306241110+155057805t+22441230t^2+935550t^3
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 48300t^4 - 5649t^5 - 182t^6 - 2t^7 + 89282088000(25+3t)e^t\big],\\
\theta_2^{(3)}(t) &= 504\big[22151115 + 6411780t + 400950t^2 - 27600t^3 - 4035t^4\\
&\quad - 156t^5 - 2t^6 + 12754584000(28+3t)e^t\big],\\
\theta_2^{(4)}(t) &= 6048\big[534315 + 66825t - 6900t^2 - 1345t^3 - 65t^4 - t^5\\
&\quad + 1062882000(31+3t)e^t\big],\\
\theta_2^{(5)}(t) &= 30240\big[13365 - 2760t - 807t^2 - 52t^3 - t^4\\
&\quad + 212576400(34+3t)e^t\big],\\
\theta_2^{(6)}(t) &= 60480\big[106288200(37+3t)e^t - 1380 - 807t - 78t^2 - 2t^3\big],\\
\theta_2^{(7)}(t) &= 181440\big[35429400(40+3t)e^t - 269 - 52t - 2t^2\big],\\
\theta_2^{(8)}(t) &= 725760\big[8857350(43+3t)e^t - 13 - t\big],\\
\theta_2^{(9)}(t) &= 725760\big[8857350(46+3t)e^t - 1\big].
\end{aligned}
$$

It is easy to calculate that

$$
\begin{aligned}
\theta'(0) &= 0, & \theta''(0) &= 0,\\
\theta^{(3)}(0) &= 0, & \theta^{(4)}(0) &= 0,\\
\theta^{(5)}(0) &= 1632960000, & \theta^{(6)}(0) &= 17635968000,\\
\theta^{(7)}(0) &= 116321184000, & \theta^{(8)}(0) &= 602017920000,\\
\theta^{(9)}(0) &= 2706957792000, & \theta^{(10)}(0) &= 11121382944000,\\
\theta_1'(0) &= 31830835680000, & \theta_1''(0) &= 83910208435200,\\
\theta_1^{(3)}(0) &= 208999489144320, & \theta_1^{(4)}(0) &= 500203983121920,\\
\theta_1^{(5)}(0) &= 1163218362768000, & \theta_1^{(6)}(0) &= 2648180949926400,\\
\theta_1^{(7)}(0) &= 5932619924353920, & \theta_1^{(8)}(0) &= 13125845965639680,\\
\theta_1^{(9)}(0) &= 28754776198995840, & \theta_1^{(10)}(0) &= 62489726878118400,\\
\theta_2'(0) &= 141434891210025, & \theta_2''(0) &= 160729807759920,\\
\theta_2^{(3)}(0) &= 180003853569960, & \theta_2^{(4)}(0) &= 199280851953120,\\
\theta_2^{(5)}(0) &= 218562955581600, & \theta_2^{(6)}(0) &= 237847398969600,\\
\theta_2^{(7)}(0) &= 257132364632640, & \theta_2^{(8)}(0) &= 276417335013120,
\end{aligned}
$$

$$\theta_2^{(9)}(0) = 295702274730240.$$

Since $\theta_2^{(9)}(t)$ is increasing, so $\theta_2^{(9)}(t) > 0$ on $(0,\infty)$, which means that $\theta_2^{(8)}(t)$ is increasing and positive on $(0,\infty)$. By the same argument, it is derived that the functions $\theta_2^{(i)}(t)$ for $1 \le i \le 9$, $\theta_1^{(i)}(t)$ and $\theta^{(i)}(t)$ for $1 \le i \le 10$ are increasing and positive on $(0,\infty)$. Therefore, the function $\theta(t)$ is increasing and positive on $(0,\infty)$, which implies that the function $H(x)$ is completely monotonic on $(0,\infty)$. Because the function $\frac{2}{x^2}$ is completely monotonic on $(0,\infty)$ and the product of finite completely monotonic functions are also completely monotonic, we obtain that the function $g(x) - g(x+1)$ is completely monotonic on $(0,\infty)$, which is equivalent to

$$(-1)^k[g(x) - g(x+1)]^{(k)} = (-1)^k g^{(k)}(x) - (-1)^k g^{(k)}(x+1) \ge 0$$

for $k \le 0$ on $(0,\infty)$. By induction, we have

$$\begin{aligned}(-1)^k g^{(k)}(x) \ge (-1)^k g^{(k)}(x+1) \ge (-1)^k g^{(k)}(x+2) \ge \cdots \\ \ge (-1)^k g^{(k)}(x+m) \ge \cdots \ge \lim_{m\to\infty}[(-1)^k g^{(k)}(x+m)] = 0\end{aligned}$$

for $k \ge 0$ on $(0,\infty)$. The proof of Theorem 1 is complete.

Remark 2. Simplification yields that the function

$$H(x) = \psi'(x) - \frac{Q(x)}{1800x^2(1+x)^{10}(2+x)^{10}} \tag{10}$$

is completely monotonic on $(0,\infty)$, where

$$\begin{aligned}Q(x) = {}& 1382400 + 21657600x + 162792960x^2 + 778137600x^3 \\ & + 2645782983x^4 + 6789381590x^5 + 13626443025x^6 \\ & + 21889330810x^7 + 28579049475x^8 + 30634381522x^9 \\ & + 27125436630x^{10} + 19896883200x^{11} + 12088287630x^{12} \\ & + 6063596590x^{13} + 2494770300x^{14} + 832958400x^{15} \\ & + 222060150x^{16} + 46134540x^{17} + 7195500x^{18} \\ & + 792300x^{19} + 54900x^{20} + 1800x^{21}\end{aligned}$$

for $x \in (0,\infty)$.

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